HL Paper 1

Part A If z is a non-zero complex number, we define $L(z)$ by the equation [9]

$$L(z) = \ln|z| + \mathrm{i}\arg(z),\ 0 \leqslant \arg(z) < 2\pi.$$

(a) Show that when z is a positive real number, $L(z) = \ln z$.

(b) Use the equation to calculate

(i) $L(-1)$;

(ii) $L(1 - \mathrm{i})$;

(iii) $L(-1 + \mathrm{i})$.

(c) Hence show that the property $L(z_1 z_2) = L(z_1) + L(z_2)$ does not hold for all values of z_1 and z_2 .

Part B Let f be a function with domain $\mathbb{R}$ that satisfies the conditions, [14]

$f(x + y) = f(x)f(y)$, for all x and y and $f(0) \neq 0$.

(a) Show that $f(0) = 1$.

(b) Prove that $f(x) \neq 0$, for all $x \in \mathbb{R}$.

(c) Assuming that $f'(x)$ exists for all $x \in \mathbb{R}$, use the definition of derivative to show that $f(x)$ satisfies the differential equation $f'(x) = k\, f(x)$, where $k = f'(0)$.

(d) Solve the differential equation to find an expression for $f(x)$.

A gourmet chef is renowned for her spherical shaped soufflé. Once it is put in the oven, its volume increases at a rate proportional to its radius.

(a) Show that the radius r cm of the soufflé, at time t minutes after it has been put in the oven, satisfies the differential equation $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{k}{r}$, where k is a constant.

(b) Given that the radius of the soufflé is 8 cm when it goes in the oven, and 12 cm when it's cooked 30 minutes later, find, to the nearest cm, its radius after 15 minutes in the oven.

Find y in terms of x, given that $(1 + x^3)\frac{\mathrm{d}y}{\mathrm{d}x} = 2x^2 \tan y$ and $y = \frac{\pi}{2}$ when $x = 0$.

A certain population can be modelled by the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = k\,y\cos kt$, where y is the population at time t hours and k is a positive constant.

(a) Given that $y = y_0$ when $t = 0$, express y in terms of k , t and y_0 .

(b) Find the ratio of the minimum size of the population to the maximum size of the population.

The curve C with equation $y = f(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{\ln y}(x+2),\ y > 1,$$

and $y = \mathrm{e}$ when $x = 2$.

a. Find the equation of the tangent to C at the point (2, e). [3]

b. Find $f(x)$. [11]

c. Determine the largest possible domain of f. [6]

d. Show that the equation $f(x) = f'(x)$ has no solution. [4]